PERIODIC SOLUTIONS OF SECOND ORDER NON-LAGRANGIAN SYSTEMS

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ABSTRACT

Using results in bifurcation theory, we show the existence of periodic solutions of a large class of non-Lagrangian systems of the form

$$u'' + A_1 v' + B_1 u + F_1(w, w', w'') = 0$$
$$v'' + A_2 u' + B_2 v + F_2(w, w', w'') = 0$$

where $w = (u, v)$.

1. Introduction

New variational techniques for the study of periodic solutions of the system of nonlinear differential equations $w'' + Bw + F(w) = 0$ were introduced by M. S. Berger [1]. These have since been used by Berger himself [2] and by others [3]. In [9] the author extended these methods and proved the existence of periodic solutions of Euler-Lagrange systems of the form $w'' + Aw' + Bw + F(w, w', w'') = 0$. This proof depends rather strongly on the Lagrangian nature of the system. In this paper we consider the not necessarily Lagrangian system

$$u'' + A_1 v' + B_1 u + F_1(w, w', w'') = 0$$
$$v'' + A_2 u' + B_2 v + F_2(w, w' \; w'') = 0$$

where $w = (u, v)$, under hypotheses replacing those of our other work. With these hypotheses, using Berger's techniques and recent results in bifurcation theory, we prove the existence of a one-parameter family of nontrivial periodic solutions $w_\delta(t)$ of period $2\pi\lambda(\delta)$ such that $w_\delta(t) \to 0$ and $2\pi\lambda(\delta) \to 2\pi\lambda_1$ as $\delta \to 0$ where $2\pi\lambda_1$ is the period of the periodic solution of the linearized system.

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2. Second order systems

We shall be concerned with the existence of families of periodic solutions for autonomous systems of ordinary differential equations of the form

$$w'' + Aw' + Bw + F(w, w', w'') = 0 \tag{1}$$

where w denotes the n vector $(w_1(t), \cdots, w_n(t))$ of real-valued functions,

$$w' = (dw_1/dt, \cdots, dw_n/dt),$$

A and B are real $n \times n$ matrices and F is a C^1 (continuously differentiable) function of $3n$ variables with

$$F(0) = 0 \text{ and } F'(0) = 0.$$

We assume that A and B are of the special form

$$A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

where A_1 is a $q \times r$ matrix, A_2 an $r \times q$ matrix, B_1 a *nonsingular* $r \times r$ matrix, and B_2 a $q \times q$ matrix with $0 \leqq q < n$ and $0 < r \leqq n$. Thus if

$$w = (u, v) = (u_1, \cdots, u_r, v_1, \cdots, v_q),$$

system (1) can be written as

$$u'' + A_1 v' + B_1 u + F_1(u, v, u', v', u'', v'') = 0, \tag{2}$$

$$v'' + A_2 u' + B_2 v + F_2(u, v, u', v', u'', v'') = 0. \tag{3}$$

Let us further assume

$$F_1(u, -v, -u', v', u'', -v'') = F_1(u, v, u', v', u'', v'')$$

and

$$F_2(u, -v, -u', v', u'', -v'') = -F_2(u, v, u', v', u'', v'').$$

This situation can arise when, for example, F is the gradient of an even function of v, u' and v''.

For subsequent use we adopt the following conventions. Let X be a Banach space. For any operator $T(\lambda, x)$ mapping a subset of $\mathbb{R} \times X$ into X, we say $\lambda \in \mathbb{R}$ is an eigenvalue of T if there exists a nonzero vector $x \in X$, called an eigenvector such that $T(\lambda, x) = x$. When $X = \mathbb{R}^n$ and for each fixed λ, $T(\lambda\ x) = T(\lambda)x$ is a linear map, and $\det(I - T(\lambda))$ is a polynomial in λ, we define the algebraic multiplicity of an eigenvalue λ, to be the multiplicity of λ as a root of the polynomial $\det(I - T(\lambda))$ and the geometric multiplicity to be the $\dim N(I - T(\lambda))$, where $N(\ )$ denotes the null space. The range of T will be denoted by $R(T)$.

By letting $x_1 = w$ and $x_2 = w'$, considering the system

$$x_1' = x_2$$
$$x_2' = -Bx_1 - Ax_2,$$

and using standard results from the theory of ordinary differential equations [3], one can readily verify that the linearized equation

$$w'' + Aw' + Bw = 0 \tag{4}$$

has a periodic solution $w(t)$ of period $2\pi\lambda_0$, λ_0 positive, if and only if λ_0 is a positive eigenvalue of the matrix operator $\lambda^2 B + i\lambda A$.

There is a relationship between the periodic solutions of (2), (3) and those of (4).

THEOREM 1. *Suppose $\lambda^2 B + i\lambda A$ has N different positive eigenvalues $\lambda_1, \cdots, \lambda_N$, satisfying $\lambda_1/\lambda_j =$ integer $(j = 1, \cdots, N)$. Suppose further that the algebraic multiplicity of each of these eigenvalues is equal to the geometric multiplicity and the sum of the multiplicities of these eigenvalues is odd. Then (2), (3) has a one-parameter family of nontrivial periodic solution $w_\delta(t) = (u_\delta(t), v_\delta(t))$ with period $2\pi\lambda(\delta)$ such that $w_\delta(t) \to 0$ and $2\pi\lambda(\delta) \to 2\pi\lambda_1$ as $\delta \to 0$.*

THEOREM 2. *If λ_1 is a positive simple eigenvalue of $\lambda^2 B + i\lambda A$ and there are no other positive eigenvalues λ satisfying $\lambda_1/\lambda =$ integer, then (2), (3) has a one-parameter family of nontrivial periodic solutions $w_\delta(t) = w(\delta, t)$ of period $2\pi\lambda(\delta)$ such that $w(\delta, t) \to 0$ and $2\pi\lambda(\delta) \to 2\pi\lambda_1$ as $\delta \to 0$. The functions $w(\delta, t)$ and $\lambda(\delta)$ are continuous in δ and t and are real analytic or $m - 1$ times continuously differentiable as F is real analytic or m times continuously differentiable.*

These theorems will be proved in Section 3 by the variational techniques developed by Berger [1] and the following results of bifurcation theory.

THEOREM 3. *Let X be a Banach space and L a bounded linear map of X into X, with an eigenvalue λ_0 such that $I - \lambda_0 L$ is a Fredholm operator of index zero, $R(I - \lambda_0 L) \cap N(I - \lambda_0 L) = \{0\}$ and $\dim N(I - \lambda_0 L)$ in odd. Suppose T is a C^1 map of a neighborhood of $(\lambda_0, 0) \in \mathbb{R} \times X$ into X such that $T(\lambda, 0) \equiv 0$. $T_x(\lambda, 0) \equiv 0$ and $T_\lambda(\lambda, x) = o(\|x\|)$ uniformly for λ near λ_0. Then λ_0 is a* *bifurcation point of the equation* $x = \lambda Lx + T(\lambda, x)$. *Moreover if* λ_0 *is a simple eigenvalue, then the bifurcating solutions of the equation are of the form* $(\lambda(\varepsilon), x(\varepsilon))$ *for* ε *in an interval about zero, where* $(\lambda(\varepsilon), x(\varepsilon))$ *is continuous and real analytic or in* C^{m-1} *as* T *is real analytic or in* C^m.

For a proof of this theorem we refer the reader to [8].

3. Proof of Theorems 1 and 2

Following Berger [1] we introduce a change of variables and set $t = \lambda s$ where λ is a real constant to be determined. The resulting system is then

$$\omega'' + \lambda A\omega' + \lambda^2 B\omega + \lambda^2 \hat{F}(\lambda, \omega) = 0 \tag{5}$$

where

$$\hat{F}(\lambda, \omega) = (\hat{F}_1(\lambda, \omega), \hat{F}_2(\lambda, \omega)) = (F_1(\omega, \lambda^{-1}\omega', \lambda^{-2}\omega''), F_2(\omega, \lambda^{-1}\omega', \lambda^{-2}\omega''))$$

and $\omega = (\mu, \nu) = (\mu_1, \cdots, \mu_r, \nu_1, \cdots, \nu_q)$. It suffices to determine the 2π periodic solutions of (5) as they correspond to the $2\pi\lambda$ periodic solutions of (2), (3).

Even 2π periodic functions are of the form $\mu = \mu_0 + \mu^*$, where μ_0 is of mean-value zero, that is, $1/2\pi\int_0^{2\pi}\mu_0(s)ds = 0$ and μ^* is equal to the mean value of μ. Thus (5) can be written in the form

$$B_1\mu^* + \frac{1}{2\pi}\int_0^{2\pi} \hat{F}_1(\lambda, \mu_0(s) + \mu^*, \nu(s))\, ds = 0 \tag{6}$$

(7)

$$\mu_0'' + \lambda A_1\nu' + \lambda^2[B_1\mu_0 + \hat{F}_1(\lambda, \mu_0 + \mu^*, \nu) - \frac{1}{2\pi}\int_0^{2\pi} \hat{F}_1(\lambda,\mu_0(s)+\mu^*,\nu(s))ds]=0$$

$$\nu'' + \lambda A_2\mu_0' + \lambda^2[B_2\nu + \hat{F}_2(\lambda, \mu_0 + \mu^*, \nu)] = 0. \tag{8}$$

To apply Theorem 3 we introduce the appropriate Banach spaces. Let $\mathscr{C}_1$ be the Banach space of twice continuously differentiable, 2π periodic, even r-vector functions of mean value zero with the usual supremum norm and $\mathscr{C}_2$ the Banach space of twice continuously differentiable, 2π periodic, odd q-vector functions with the supremum norm. Then set $\mathscr{C} = \mathscr{C}_1 \times \mathscr{C}_2$. Let H_1 be the Hilbert space of absolutely continuous, 2π periodic, even r-vector functions of mean value zero such that $\int_0^{2\pi}(\mu_0'(s))^2\, ds < \infty$ and H_2 the Hilbert space of absolutely continuous, 2π periodic, odd q-vector functions such that $\int_0^{2\pi}(\nu'(s))^2 ds < \infty$. Let $H = H_1 \times H_2$ with inner product defined by $\langle x, y\rangle = \int_0^{2\pi}x'(s)\cdot y'(s)\, ds$, $x = (\mu_0, \nu)$. And lastly let X^* be the space of the μ^* with Euclidean norm.

Finding solutions of (6), (7), (8) is therefore equivalent to finding solutions of the operator equations

$$B_1\mu^* + \frac{1}{2\pi}\int_0^{2\pi} \hat{F}_1(\lambda, x(s) + \mu^*)\,ds = 0$$

$$x - \lambda\mathscr{A}x - \lambda^2\mathscr{B}x - \lambda^2\mathscr{F}(\lambda, x + \mu^*) = 0$$

in $(\mathscr{C} \times X^*) \cap (H \times X^*)$ where $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{F}$ are defined implicitly by

$$\langle \mathscr{A}x, y\rangle = \int_0^{2\pi} Ax'(s)\cdot y(s)ds,\ \langle \mathscr{B}x, y\rangle = \int_0^{2\pi} Bx(s)\cdot y(s)\,ds,$$

and

$$\langle \mathscr{F}(\lambda, x + \mu^*), y\rangle = \int_0^{2\pi} \hat{F}(\lambda, x(s) + \mu^*)\cdot y(s)\,ds.$$

Since $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{F}$ satisfy the equations $d\mathscr{A}x/ds = -Ax$, $d^2\mathscr{B}x/ds^2 = -Bx$ and $\partial^2\mathscr{F}(\lambda, x + \mu^*)/\partial s^2 = -\hat{F}(\lambda, x + \mu^*)$, an application of the Ascoli-Arzela theorem will verify that $\mathscr{A}$ and $\mathscr{B}$ are compact maps of $\mathscr{C}$ into $\mathscr{C}$ and that $(\mathscr{F}, 1/2\pi \int_0^{2\pi}\hat{F}_1)$ is a continuously differentiable nonlinear map of $\mathbb{R} \times (\mathscr{C} \times X^*)$ into $\mathscr{C} \times X^*$ satisfying the hypotheses of Theorem 3.

By hypothesis, B_1 is nonsingular. Hence by the implicit function theorem [4, p. 194], [8] there exists a C^1 function f such that

$$B_1 f(\lambda, x) + \frac{1}{2\pi}\int_0^{2\pi} \hat{F}_1(\lambda, x(s) + f(\lambda, x(s)))\,ds \equiv 0$$

for (λ, x) near $(\lambda_0, 0)$. In addition $f \in C^m$ or real analytic as $F \in C^m$ or real analytic. Thus it suffices to find solutions of

$$x = \lambda\mathscr{A}x + \lambda^2\mathscr{B}x + \lambda^2\mathscr{F}(\lambda, x + f(\lambda, x)). \tag{9}$$

As (9) does not satisfy the hypotheses of Theorem 3 we consider instead the system

$$z = \lambda Lz + T(\lambda, z) \tag{10}$$

where $z = (y, x) \in \mathscr{C} \times \mathscr{C}$ and

$$L = \begin{pmatrix} 0 & \mathscr{B} \\ I & \mathscr{A} \end{pmatrix} \text{and } T(\lambda, z) = T(\lambda, y\ x) = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{F}(\lambda, x + f(\lambda, x)) \end{pmatrix}.$$

Clearly any solution of (10) will give us a solution of (9) and conversely, any solution of (9) gives a solution of (10). Thus to complete the proof it suffices to show that L satisfies the hypotheses of Theorem 3.

If λ_k is a positive eigenvalue of $\lambda^2 B + i\lambda A$, λ_1/λ_k is an integer, and $(a + ib, c + id)$ with $a, b \in \mathbb{R}^r$ and $c, d \in \mathbb{R}^q$ is an eigenvector corresponding to λ_k, then by equating components we have the equalities

$$\begin{aligned} \lambda_k^2 B_1 a - \lambda_k A_1 d &= a \\ \lambda_k^2 B_2 d + \lambda_k A_2 a &= d \\ \lambda_k^2 B_1 b + \lambda_k A_1 c &= b \\ \lambda_k^2 B_2 c - \lambda_k A_2 b &= c. \end{aligned}$$

Hence $N(I - \lambda_k^2 B - i\lambda_k A)$ is the span of vectors of the form $(a_{n_k}^{(j)}, id_{n_k}^{(j)})$ where $n = \lambda_1/\lambda_k$ and $j = 1, \cdots, r_k$, r_k the multiplicity of λ_k. Thus by standard results from the theory of ordinary differential equations [3], the set

$$\{(a_{n_k}^{(j)} \cos n_k s,\ d_{n_k}^{(j)} \sin n_k s))\},\ j = 1, \cdots r_k,\ k = 1, \cdots, N$$

is a basis of $N(I - \lambda_1 \mathscr{A} - \lambda_1^2 \mathscr{B})$. Consequently

$$\dim N(I - \lambda_1 L) = \dim N(I - \lambda_1 \mathscr{A} - \lambda_1^2 \mathscr{B}) = \sum_{j=1}^{N} \dim N(I - \lambda_j^2 B - i\lambda_j A)$$

is odd and is equal to one if λ_1 is as in Theorem 2.

Now we show $R(I - \lambda_1 L) \cap N(I - \lambda_1 L) = \{0\}$. Let $E = \{\lambda_1/\lambda_1, \cdots, \lambda_1/\lambda_N\}$. Suppose $z = (y, x) \in \mathscr{C} \times \mathscr{C}$ and $(I - \lambda_1 L)z \in N(I - \lambda_1 L)$. Then $y = \alpha + \beta$ and $x = \gamma + \delta$ where

$$\begin{aligned} \alpha &= \sum_{j=1}^{N} (a_j \cos n_j s,\ a_j' \sin n_j s), \quad \text{for } n_j \in E \\ \beta &= \sum_{m=2}^{\infty} (b_m \cos ms,\ b_m' \sin ms), \quad \text{for } m \notin E \\ \gamma &= \sum_{j=1}^{N} (c_j \cos n_j s,\ c_j' \sin n_j s), \quad \text{for } n \in E \\ \delta &= \sum_{m=2}^{\infty} (d_m \cos ms,\ d_m' \sin ms), \quad \text{for } m \notin E. \end{aligned}$$

Since $I - \lambda_1 L$: $((a \cos ms, b \sin ms), (c \cos ms, d \sin ms)) \to ((a' \cos ms, b' \sin ms), (c' \cos ms, d' \sin ms))$ for $m = 1, 2, \cdots$, it follows that

$$(I - \lambda_1 L)(\beta, \delta) = 0.$$

On the other hand $(I - \lambda_1 L)(\alpha, \gamma) \in N(I - \lambda_1 L)$ and $(I - \lambda_1 L)(\alpha, \gamma) \neq 0$ implies that

$$(I - \lambda_1 Q_j)(a_j, ia'_j, c_j, ic'_j) \in N(I - \lambda_1 Q_j)$$

and

$$(I - \lambda_1 Q_j)(a_j, ia'_j, c_j, ic'_j) \neq 0$$

where

$$Q_j = \begin{pmatrix} 0 & n_j^{-2} B \\ I & in_j^{-1} A \end{pmatrix}$$

$j = 1, \cdots, N$. However this is impossible by our hypotheses.

Indeed, since $\det(I - \lambda_1 Q_j) = \det(I - \lambda_j^2 B - i\lambda_j A)$ the algebraic multiplicity of λ_1 equals its geometric multiplicity. Hence by the Jordan canonical form of Q_j it readily follows $N(I - \lambda_1 Q_j) \cap R(I - \lambda_1 Q_j) = \{0\}$.

Now we verify $I - \lambda_1 L$ is a Fredholm operator on $\mathscr{C} \times \mathscr{C}$. To show $R(I - \lambda_1 L)$ is closed, it suffices to prove it maps bounded, closed sets into closed sets [5, p. 99]. Suppose D is a bounded closed set and

$$\{(I - \lambda_1 L)z_n\} = \{(y_n - \lambda_1 \mathscr{B} x_n, x_n - \lambda_1(\mathscr{A} x_n + y_n))\}$$

is a Cauchy sequence. By compactness there is a subsequence such that $\{\lambda_1 \mathscr{B} x_n\}$ and $\{\lambda_1 \mathscr{A} x_n\}$ are Cauchy sequences and therefore $y_n \to y$. This in turn implies $\{x_n\}$ is a Cauchy sequence and $x_n \to x$. Thus by continuity $(I - \lambda_1 L)z_n \to (I - \lambda_1 L)(y, x)$.

Next we show that $I - \lambda_1 L$ is of index zero. A simple computation will verify that the operator

$$L^* = \begin{pmatrix} 0 & I \\ \mathscr{B}^* & \mathscr{A}^* \end{pmatrix}$$

is the adjoint of L, where $\mathscr{B}^*$ and $\mathscr{A}^*$ are the adjoints of $\mathscr{B}$ and $\mathscr{A}$. By the Fredholm alternative theorem [7, p. 219]

$$\begin{aligned} \dim N(I - \lambda_1 L) &= \dim N(I - \lambda_1(\mathscr{A} + \lambda_1 \mathscr{B})) \\ &= \dim N(I - \lambda_1(\mathscr{A}^* + \lambda_1 \mathscr{B}^*)) = \dim N(I - \lambda_1 L^*). \end{aligned}$$

Thus as we have verified that (10) satisfies the hypotheses of Theorem 3, Theorems 1 and 2 follow from Theorem 3.

REMARK. Suppose (1) has the form

$$w'' + Bw + F(w, w', w'') = 0 \tag{11}$$

where B is a nonzero $n \times n$ matrix and $F(-w, w', -w'') = -F(w, w', w'')$. Then if $\mathscr{C}$ and H are replaced by $\mathscr{C}_2$ and H_2 the proof of Theorems 1 and 2 will show that these theorems are also true for the system (11).

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